Microeconomic Theory II
PS1 - Solution Topics

1.
(a) Independence axiom verification:

$$\begin{aligned} p &\sim q \iff \sum_{j=1}^{n} p(x_j)\sqrt{x_j} = \sum_{j=1}^{n} q(x_j)\sqrt{x_j} \\ &\iff \alpha\sum_{j=1}^{n} p(x_j)\sqrt{x_j} + (1-\alpha)\sum_{j=1}^{n} r(x_j)\sqrt{x_j} = \alpha\sum_{j=1}^{n} q(x_j)\sqrt{x_j} + (1-\alpha)\sum_{j=1}^{n} r(x_j)\sqrt{x_j} \\ &\iff \sum_{j=1}^{n}\left(\alpha p(x_j)\sqrt{x_j} + (1-\alpha) r(x_j)\sqrt{x_j}\right) = \sum_{j=1}^{n}\left(\alpha q(x_j)\sqrt{x_j} + (1-\alpha) r(x_j)\sqrt{x_j}\right) \\ &\iff \alpha p + (1-\alpha) r \sim \alpha q + (1-\alpha) r \end{aligned}$$

(b) Since $V(p) = [U(p)]^2$ the utility function V is a strictly increasing transformation of U, and hence V and U represent the same preferences. Thus, the independence axiom is also valid for V.

Independence axiom verification:

$$\begin{aligned} p &\sim q \iff \left[\sum_{j=1}^{n} p(x_j)\sqrt{x_j}\right]^2 = \left[\sum_{j=1}^{n} q(x_j)\sqrt{x_j}\right]^2 \iff \sum_{j=1}^{n} \alpha p(x_j)\sqrt{x_j} = \sum_{j=1}^{n} \alpha q(x_j)\sqrt{x_j} \\ &\iff \sum_{j=1}^{n} \alpha p(x_j)\sqrt{x_j} + \sum_{j=1}^{n}(1-\alpha) r(x_j)\sqrt{x_j} = \sum_{j=1}^{n} \alpha q(x_j)\sqrt{x_j} + \sum_{j=1}^{n}(1-\alpha) r(x_j)\sqrt{x_j} \\ &\iff \left[\sum_{j=1}^{n}(\alpha p(x_j) + (1-\alpha) r(x_j))\sqrt{x_j}\right]^2 = \left[\sum_{j=1}^{n}(\alpha q(x_j) + (1-\alpha) r(x_j))\sqrt{x_j}\right]^2 \\ &\iff \alpha p + (1-\alpha) r \sim \alpha q + (1-\alpha) r. \end{aligned}$$

(c) The Bernoulli utility function is $u(x) = \sqrt{x}$ and, hence, the certainty equivalent CE is given by

$$\sqrt{CE} = \frac{2\sqrt{x_1} + \sqrt{x_2}}{3} \iff CE = \left(\frac{2\sqrt{x_1} + \sqrt{x_2}}{3}\right)^2.$$

2. Her final wealth will be either $Y + w$ or $Y - w$. Hence she solves,

$$\begin{aligned} \max_w \{pu(Y+w) + (1-p)u(Y-w)\} &= \max_w \left\{-p.e^{-r(Y+w)} - (1-p).e^{-r(Y-w)}\right\} \\ FOC &: (1-p)\,e^{rw} = pe^{-rw} \end{aligned}$$

Hence,

$$w^* = \frac{1}{2r}\ln\left(\frac{p}{1-p}\right).$$

Checking the concavity of the objective function in order to ensure that the solution is a maximum,

$$SOC: -r^2 p.e^{-r(Y+w)} - r^2(1-p).e^{-r(Y-w)} < 0, \forall w.$$

Note that a positive wager will be made for $p > 1/2$. The wager decreases as the risk coefficient increases. Note also that in this case the results is independent of the initial wealth - a particular feature of this utility function.

3.

(a) Property (i) is equivalent to (ii). Define $F(u_2)$ implicitly by $u_1(w) = F(u_2(w))$. Note that monotonicity of the utility functions implies that F is well defined i.e., that there is a unique value of $F(u_2)$ for each value of u_2. Differentiate this definition twice to find

$$\begin{aligned} u_1'(w) &= F'(u_2).u_2'(w) \\ u_1''(w) &= F''(u_2).(u_2'(w))^2 + F'(u_2).u_2''(w). \end{aligned}$$

Since $u_1'(w) > 0$ and $u_2'(w) > 0$, the first equation establishes $F'(u_2) > 0$. Dividing the second equation by the first gives us

$$\frac{u_1''(w)}{u_1'(w)} = \frac{F''(u_2)}{F'(u_2)}.u_2'(w) + \frac{u_2''(w)}{u_2'(w)}.$$

Rearranging gives us

$$\frac{F''(u_2)}{F'(u_2)}.u_2'(w) = \frac{u_1''(w)}{u_1'(w)} - \frac{u_2''(w)}{u_2'(w)}. \tag{1}$$

We can prove now that property (i) implies (ii) since from (i) the RHS of (1) is negative, hence this shows that $F''(u_2) < 0$ as required by property (ii). On the other hand, we can prove that (ii) implies (i) since from (ii) and the fact that $u_2'(w) > 0$ by assumption, the LHS of (1) is negative and so $\frac{u_1''(w)}{u_1'(w)} - \frac{u_2''(w)}{u_2'(w)} < 0 \iff -\frac{u_1''(w)}{u_1'(w)} > -\frac{u_2''(w)}{u_2'(w)}$ and we are done.

(b) Property (ii) is equivalent to (iii).

■P.(ii) implies P.(iii). This follows from the following chain of inequalities:

$$\begin{aligned} u_1(w - \pi_{u_1}) &= E[u_1(w+\tilde{\epsilon})] = E[F(u_2(w+\tilde{\epsilon}))] \\ &< F(E[(u_2(w+\tilde{\epsilon}))]) = F(u_2(w-\pi_{u_2})) = u_1(w-\pi_{u_2}) \end{aligned}$$

All of these relationships follow from the definition of the risk premium except for the inequality, which follows from Jensen's inequality. Comparing the first and the last terms, we see that $\pi_{u_1} > \pi_{u_2}$.

■P.(iii) implies P.(ii). It was already shown (in (a)) that F is an increasing function. Now,

$$\begin{aligned} \pi_{u_1} &> \pi_{u_2} \iff u_1(w-\pi_{u_1}) < u_1(w-\pi_{u_2}) \iff E[u_1(w+\tilde{\epsilon})] < F(u_2(w-\pi_{u_2})) \\ &\iff E[F(u_2(w+\tilde{\epsilon}))] < F(E[(u_2(w+\tilde{\epsilon}))]), \end{aligned}$$

where the last inequality implies F to be a concave function and we are done.

4.

(a) True. Just let $u(x) = x$ in the definition of first-order stochastic dominance.

(b) False. Suppose there are three outcomes: 1, 2, 3. Compare $L_G = (0; 1; 0)$ with $L_F = (0.4; 0; 0.6)$. The latter has a higher mean, but does not first-order stochastically dominate the former.

(c)

True. Let $u(x) = x^2$ (a non-decreasing convex function of x). Using a convex function allows us to invert the definition of second-order stochastic dominance:

$$\int x^2 dF(x) \leq \int x^2 dG(x)$$

Note that $\left(\int xdF(x)\right)^2 = \left(\int xdG(x)\right)^2$ and subtract it from both sides of the inequality above. We have:

$$\int x^2 dF(x) - \left(\int xdF(x)\right)^2 \leq \int x^2 dG(x) - \left(\int xdG(x)\right)^2$$

(d) False. Suppose there are five outcomes: 1, 2, 3, 4, 5. Compare $L_G = (0.01; 0; 0.98; 0; 0.01)$ with $L_F = (0; 0.5; 0; 0.5; 0)$. They have the same mean. G has a smaller variance than F, but it does not second-order dominate it. To see this, take:

$$u(x) = \begin{cases} 2 & if\ x > 2 \\ x & if\ x \leq 2 \end{cases}$$

We have:

$$\begin{aligned} \int u(x)dF(x) &= 2 \\ \int u(x)dG(x) &= 1.99 \end{aligned}$$

5.

(a)

	L	R
U	1, 0	0, 1
D	0, 1	1, 0

In this game, there is no Nash equilibrium in pure strategies, but we can find a sequence of justifications such that U is rationalizable for 1: [U,L,D,R,U,L,...].

(b)

	L	R
U	1, 1	1, 1
D	1, 1	0, 0

(U,L); (U,R); and (D,L) are NE.

(c)

	L	R
U	$1,0$	$0,1$
D	$0,1$	$1,1$

The game has a unique NE in pure strategies - (D,R) -, but there are no strictly dominant strategies.

(d)

	L	R
U	$1,0$	$0,1$
D	$0,1$	$1,0$

Here, there are no NE in pure strategies. However:

- For player 2 to randomize, he must be indifferent between L and R. Suppose that player 1 chooses U with probability p. Hence:

$$\underbrace{p \times 0 + (1-p) \times 1}_{\text{If player 2 plays L}} = \underbrace{p \times 1 + (1-p) \times 0}_{\text{If player 2 plays R}} \Rightarrow p = \frac{1}{2}$$

- For player 1 to randomize, he must be indifferent between U and D. Suppose that player 2 chooses L with probability q. As this is a symmetric game, it must be that $q = \frac{1}{2}$.

(e) We now need to describe the values of the parameters such that the game has two NE in pure strategies.

Let us look for parameter values such that (U,L) and (D,R) are the only NE in pure strategies.

- For (U,L) to be a NE, we need $a \geq e$ and $b \geq d$.
- For (D,R) to be a NE, we need $g \geq c$ and $h \geq f$.
- For (U,R) not to be a NE, either $a > e$ or $h > f$.
- For (D,L) not to be a NE, either $b > d$ or $g > c$.

So, we can re-write the condition as:$a \geq e$, $b \geq d$, $g \geq c$, $h \geq f$, $a+h > e+f$ and $b+g > c+d$.

The case where (U,R) and (D,L) are the only NE is analogous.

Now, let us look at (U,L) and (U,R) as the only NE.

Clearly, player 2 must be indiferent between L and R if player one play U. Therefore, $b = d$.

Besides, so that one does not have profitable deviations, $a \geq e$ and $c \geq g$ should hold. To knock out (D,L) and (D,R) as possible NE we need to add that:

- If $a = e$, then $h > f$ and $c > g$.
- If $c = g$, then $f > h$ and $a > e$.

So, we can re-write the conditions as: $b = d$, $a \geq e$, $c \geq g$, $a + c > e + g$, $h > f$ if $a = e$, and $f > h$ if $c = g$.

The cases where (D,L) and (D,R), (U,L) and (D,L), and (U,R) and (D,R) are analogous.

(f) Best response correspondence for player 1:

	L	R
U	a	c
D	e	g

For each mixed strategy for player 2 $(q, 1-q)$, where $q \in [0, 1]$, player 1 will play U if

$$qa + (1-q)c > qe + (1-q)g \Leftrightarrow q(a-e) + (1-q)(c-g) > 0$$

And will play D if

$$q(a-e) + (1-q)(c-g) < 0$$

And will randomize if:

$$q(a-e)+(1-q)(c-g)=0$$

Therefore, player 1 only needs to know $a-e$ and $c-g$ in order to derive his best response.

6.

(a) Let x denote the location of vendor i and y the one of vendor j. Given x, y the demand for

vendors i, j's ice-cream is given by

$$d_i(x,y)=\begin{cases}\frac{x+y}{2} & \text{if } x<y\\ \frac{1}{2} & \text{if } x=y\\ 1-\frac{x+y}{2} & \text{if } x>y\end{cases} \qquad d_j(y,x)=\begin{cases}\frac{x+y}{2} & \text{if } y<x\\ \frac{1}{2} & \text{if } x=y\\ 1-\frac{x+y}{2} & \text{if } y>x\end{cases}$$

Clearly, $x=y=1/2$ is a Nash equilibrium, since if $x=1/2$, then j's best response is to set $y=1/2$ and similarly for i. To see that this is the unique Nash equilibrium, notice that in any equilibrium we must have $x=y$, since otherwise one vendor could deviate by $\epsilon>0$ and increase its sales. But, if $x=y\neq 1/2$, then again there exists a deviation for each vendor that he can increase his sales above 1/2.

(b) Let z denote the location of vendor k. Then the demand for each vendor is given by

$$\begin{aligned}
d_i(x,y,z) &= \begin{cases}\frac{x+\min(y,z)}{2} & \text{if } x<\min(y,z)\\ \frac{x+\max(y,z)}{2}+\frac{x+\min(y,z)}{2} & \text{if } \min(y,z)<x<\max(y,z)\\ \frac{1}{3} & \text{if } x=y=z\\ 1-\frac{x+\max(y,z)}{2} & \text{if } x>\max(y,z)\end{cases}\\
d_j(x,y,z) &= \begin{cases}\frac{y+\min(x,z)}{2} & \text{if } y<\min(x,z)\\ \frac{y+\max(x,z)}{2}+\frac{y+\min(x,z)}{2} & \text{if } \min(x,z)<y<\max(x,z)\\ \frac{1}{3} & \text{if } x=y=z\\ 1-\frac{y+\max(x,z)}{2} & \text{if } y>\max(x,z)\end{cases}\\
d_k(x,y,z) &= \begin{cases}\frac{z+\min(y,x)}{2} & \text{if } z<\min(x,y)\\ \frac{z+\max(y,x)}{2}+\frac{z+\min(y,x)}{2} & \text{if } \min(x,y)<z<\max(x,y)\\ \frac{1}{3} & \text{if } x=y=z\\ 1-\frac{z+\max(y,x)}{2} & \text{if } z>\max(x,y)\end{cases}.
\end{aligned}$$

Here again, if a pure strategy equilibrium existed, it would require all vendors to locate in the same location, for otherwise one of them would always have an incentive to deviate. But if they all locate at the same location, then again one of them could deviate and get a higher pay-off. Thus, there cannot exist an equilibrium in pure strategies.

7.

Under the conditions in MWG prop. 8.D.3, we need to show that any 2 person symmetric game (where $S_1=S_2$ and $u_i(s_i,s_j)=u_j(s_j,s_i)$ has a symmetric NE.

A symmetric NE is $\sigma^*=(\sigma_1^*,\sigma_2^*)$ s.t. $\sigma_1^*=\sigma_2^*$. Let $\tilde{\Sigma}=\tilde{\Sigma}_1=\tilde{\Sigma}_2$ denote the mixed strategy spaces (the same for both players because of symmetry). Let $r:\tilde{\Sigma}\rightrightarrows\tilde{\Sigma}$, where $r(\tilde{\sigma})=\arg\max_{\sigma_1'} u_1(\sigma_1',\tilde{\sigma})$, be player 1's best response correspondence. Since the game is symmetric, 2's best response correspondence will be the same.

We just need to check if the conditions of Kakutani are satisfied:

(1) Mixed strategy space is the set of probability distributions over the pure strategies ans it's a simplex - non-empty, compact, convex subset of $\mathbb{R}^{\#S_1 - 1}$.

(2) We know that $u(.)$ is continuous and the Weierstrass theorem is enough to ensure that a maximum exits (we are maximizing a continuous function on a compact set). So $r(\tilde{\sigma})$ is non-empty.

(3) $r(\tilde{\sigma})$ is convex because the set of maximizers of a quasiconcave function ($u(.)$) on a convex set $\tilde{\Sigma}$ is convex.

(4) $r(\tilde{\sigma})$ has a closed graph which follows from the continuity of $u(.)$.

We can then apply Kakutani's theorem and $\exists \tilde{\sigma} |\ \tilde{\sigma}' \in r(\tilde{\sigma}^*) \Rightarrow (\tilde{\sigma}_1^*, \tilde{\sigma}_2^*)$ is a symmetric Nash Equilibrium.