Microeconomic Theory II
PS2 - Solution Topics

1)
Each subgame has two Nash equilibria in pure strategies: (S, S), (R, R). Therefore, there are 4 SPE in pure strategies. To describe these, let us first name the strategies for players. Player 1 has 8 strategies: GSS, GSR, GRS, GRR, NSS, NSR, NRS, and NRR, where the first letter indicates whether he gives gift (G) or not (N), and the second and third letters indicate the actions he takes in case of gift and no gift, respectively. Similarly, player 2 has four strategies: SS, SR, RS, and RR. The subgame perfect equilibria are: (NSS, SS), (NRR, RR), (NRS, RS), and (GSR, SR).

2) In the second stage of the game, it is clear that each type will always play the static best response: $a = 2$ type will fight and $a = -1$ type will accommodate. Then for the first stage, with $\pi = 0.9$, the entrant will always enter since

$$EU(Enter) = 2\pi - 1 = 0.8 > EU(out) = 0,$$

and $a = 2$ type fights and $a = -1$ type accommodates.

The PBE is $\{(F, A; F, A), (E, O, E)\}$ - here we are denoting the strategy for the incumbent as $(t_{11}, t_{21}; t_{12}, t_{22})$, where t_{ij} indicates the action taken by type i in the stage j, and for the entrant as (a_1, a_{21}, a_{22}), where a_1 refers to the first information set at $t = 1$, and the other two refer to the information sets at $t = 2$. Player 2's beliefs are updated such that we have $\mu(a = -1|A) = 1$ at a_{22}, and $\mu(a = 2|F) = 1$ at a_{21}.

3)
(a) To begin with, let's find the subgame perfect Nash Equilibrium of the stage game. We can do this by backward induction. The long-run firm, observing the short-run firms's quantity x_t, chooses its quantity y_t to maximize its profit

$$\pi_t^L = y_t \left(1 - (x_t + y_t)\right).$$

Solving the first order condition, the optimum is determined as

$$y_t^* \left(x_t^*\right) = \frac{1 - x_t^*}{2}.$$

The short-run firm chooses its quantity x_t to maximize its profit

$$\pi_t^S = x_t \left(1 - (x_t + y_t)\right)$$

knowing that if it chooses x_t, the long-run firm reacts with

$$y_t^* = \frac{1 - x_t}{2}.$$

Therefore, the short-run firm's objective function can be written as a function of x_t;

$$\pi_t^S = x_t \left(1 - \left(x_t + \frac{1 - x_t}{2}\right)\right).$$

Solving the first order condition, the optimum is $x_t^* = 1/2$. Therefore, the subgame perfect Nash Equilibrium of the stage game is

$$x_t^* = \frac{1}{2},\ y_t^* = \frac{1-\frac{1}{2}}{2} = \frac{1}{4}.$$

Now let's solve for the subgame perfect Nash Equilibrium of the finitely repeated game. Since it is a finite horizon game with perfect information, we can use backward induction. At the last period, $t = T$, the players don't care about the future, and they concern only about the payoffs of that period. Therefore they must play the subgame perfect Nash equilibrium of the stage game, regardless of what happened in the past. At time $t = T - 1$, players know that their actions today don't affect tomorrow's outcome, so they will concern only about the payoff of that period. Therefore, they again must play the subgame perfect Nash equilibrium of the stage game. We can repeat this argument until we reach the first period. Therefore, the subgame perfect Nash equilibrium of the finitely repeated game is

$$x_t^* = \frac{1}{2},\ y_t^* = \frac{1}{4} \text{ for all } t,$$

regardless of history.

(b)
Consider the following trigger strategy.
Long-run firm
1) Start with playing the following strategy (*):

$$y_t(x_t) = \begin{cases} 1/2 & \text{if } x_t \leq 1/2 \\ 1 - x_t & \text{if } x_t > 1/2 \end{cases}$$

Keep playing this strategy as long as it has not deviated from it.
2) Play $y_t(x_t) = \frac{1-x_t}{2}$ if it has deviated from (*) at least once.

Short-run firms
1) Play

$$x_t = \begin{cases} 1/4 & \text{if the long-run firm has never deviated from (*) before} \\ 1/2 & \text{if the long-run firm has deviated from (*) at least once.} \end{cases}$$

To see this is actually a subgame perfect Nash equilibrium, let's check incentive to deviate. First, consider the long-run firm's incentive when it has never deviated from (*) before.

Case 1: $x_t \leq 1/2$
If it follows the strategy and chooses $y_t = 1/2$, the present period profit of the long-run firm is $\frac{1}{2}\left(1 - \left(\frac{1}{2} + x_t\right)\right)$.
Starting from the next period, the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, and the long-run firm's per period profit is 1/8, and therefore the present discounted value of the profit stream is $\frac{1}{2}\left(1 - \left(\frac{1}{2} + x_t\right)\right) + \frac{\delta}{8(1-\delta)}$.
If it is to deviate, the best it can do today is play $y_t = \frac{1-x_t}{2}$ and get the payoff of $\frac{(1-x_t)^2}{4}$.
However, starting from next period the outcome will be $x_t = 1/2$ and $y_t = 1/4$ and the long-run firm's per period profit is 1/16. Therefore the present discounted value of the profit stream is $\frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)}$.

If $\delta = 0.99$,

$$\frac{1}{2}\left(1-\left(\frac{1}{2}+x_t\right)\right)+\frac{\delta}{8\left(1-\delta\right)} > \frac{\left(1-x_t\right)^2}{4}+\frac{\delta}{16\left(1-\delta\right)},$$

and therefore it is better to follow the equilibrium strategy than to deviate.

Case 2: $x_t > 1/2$

If it follows the strategy, then $p_t = 0$, and today's payoff is 0, and the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, starting the next period. The long-run firm's per period profit is 1/8, and therefore the present discounted value of the profit sequence is $\frac{\delta}{8(1-\delta)}$.

If it is to deviate, the best it can do today is play $y_t = \frac{1-x_t}{2}$, and get the payoff of $\frac{(1-x_t)^2}{4}$. However, starting from next period, the outcome will be $x_t = 1/2$ and $y_t = 1/4$, and the long-run firm's per period profit is 1/16. Therefore the present discounted value of the profit stream is $\frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)}$.

This value is the largest when $x_t = 1/2$, and is equal to $\frac{1}{16} + \frac{\delta}{16(1-\delta)}$. If $\delta = 0.99$, it is better to follow the equilibrium strategy path than to deviate.

Second, consider the long-run firm's incentive when it has deviated from (*) before.

According to the strategy profile, future outcomes don't depend on today's behavior. Therefore the long-run firm cares only about its payoff today. Actually, by following the strategy, it is taking best response to the short-run firm.

Finally, consider the short-run firm's incentives. Since they never care about future payoff, it must be playing a best response to the long-run firm's strategy, which is actually true.

c)

In the equilibrium we saw in part (b), the per period profits on the equilibrium path were 1/8 for the long-run firm and 1/16 for the short-run firms. If there is a subgame perfect Nash equilibrium where $x_t = y_t = 1/4$ on the equilibrium path, then the per period profits on the equilibrium path are 1/8 for the long-run firm and 1/8 for the short-run firm.

Construct the following trigger strategy, which is different from part (b) only in (*) where $x_t = 1/4$:

Long-run firm

1) Start with playing the following strategy (*):

$$y_t\left(x_t\right) = \begin{cases} 1/4 & \text{if } x_t = 1/4 \\ 1/2 & x_t \le 1/2 \text{ and } x_t \ne 1/4 \\ 1/4 & \text{if } x_t > 1/2 \end{cases}$$

Keep playing this strategy as long as it has not deviated from it.

2) Play $y_t^*\left(x_t\right) = \frac{1-x_t}{2}$, if it has deviated from (*) at least once.

Short-run firms

1) Play $x_t = 1/4$ if the long-run firm has not deviated from (*) before.

2) Play $x_t = 1/2$ if the long-run firm has deviated from (*) at least once.

The incentive of the long-run firm when it is supposed to play (*) and $x_t = 1/4$ is satisfied because it gets current payoff of 1/8, which is the same as part (b), and what it can get by deviating is the same as part (b). Thus, the incentive problem of the long-run firm is the same as in part (b).

The incentive of the short-run firms is also satisfied because the payoff from following the strategy is larger than in part (b), and the payoff when deviating is the same as in part (b).

4) We construct a Bayesian Nash equilibrium (x_1^*, x_2^*), which will be in the form of $x_i^*(\theta_i) = a + b\sqrt{\theta_i}$. The expected payoff of i from investment x_i is

$$U(x_i, \theta_i) = E\left[\theta_i x_i x_j^* - x_i^3\right] = \theta_i x_i E\left[x_j^*\right] - x_i^3$$

and $x_i^*(\theta_i)$ satisfies the first order condition

$$\frac{\partial U(x_i, \theta_i)}{\partial x_i} = 0 \Longleftrightarrow \theta_i E\left[x_j^*\right] - 3x_i^2 = 0 \Longleftrightarrow x_i^* = \sqrt{\frac{\theta_i E\left[x_j^*\right]}{3}}.$$

That is, $a = 0$, and the equilibrium is in the form of $x_i^*(\theta_i) = b\sqrt{\theta_i}$ where

$$b = \sqrt{\frac{E\left[x_j^*\right]}{3}}.$$

But $x_j^* = b\sqrt{\theta_j}$, hence

$$E\left[x_j^*\right] = E\left[b\sqrt{\theta_j}\right] = bE\left[\sqrt{\theta_j}\right] =^1 \frac{2b}{3}.$$

Substituting this in the previous equation we obtain

$$b^2 = \frac{E\left[x_j^*\right]}{3} = \frac{\frac{2b}{3}}{3} \Longleftrightarrow b = \frac{2}{9}.$$

In summary,

$$x_i^* = \frac{2}{9}\sqrt{\theta_i}.$$

5) There is a unique perfect Bayesian Nash equilibrium in this game. Clearly, 1 must exit at the beginning and 2 has to go in on the right branch as he does not have any choice. The behavior at the nodes in the bottom layer is given by sequential rationality as in the figure below. Write α for the probability that 2 goes in in the center branch, β for the probability that 3 goes right, and μ for the probability 3 assigns to the center branch. In equilibrium, 3 must mix (i.e., $\beta \in (0,1)$). Because if 3 goes left, then 2 must exit at the center branch, hence 3 must assign probability 1 to the node at the right (i.e., $\mu = 0$), and hence she should play right – a contradiction. Similarly, if 3 plays right, then 2 must go in at the center branch. Given his prior beliefs (.4 and .1), $\mu = 4/5$, hence 3 must play left – a contradiction again. In order 3 to mix, she must be indifferent, i.e.,

$$1 = 0\mu + 3(1-\mu),$$

hence,

$$\mu = 2/3.$$

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[1] $E\left[\sqrt{\theta_j}\right] = \int_0^1 \sqrt{\theta_j}.1d\theta_j = \left[\frac{\theta_j^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}$ since θ_j is uniformly distributed on the unit interval.

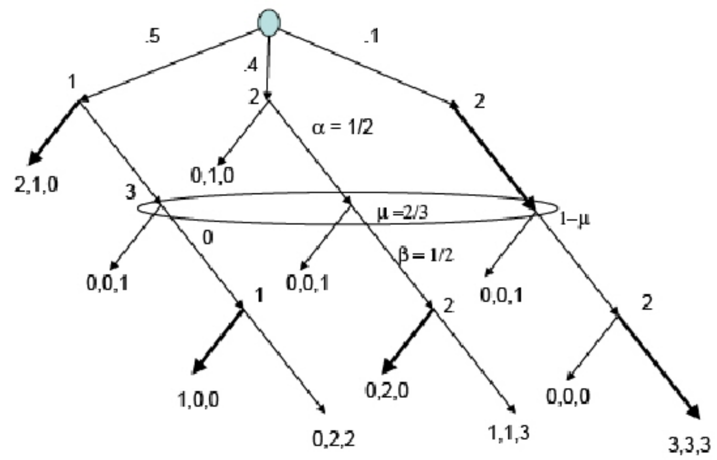
By the Bayes' rule, we must have

$$\mu = \frac{0.4\alpha}{0.4\alpha + 0.1} = \frac{2}{3} \Longleftrightarrow \alpha = \frac{1}{2}.$$

That is player 2 must mix on the center branch, and hence she must be indifferent,

$$1 = 2\beta \Longleftrightarrow \beta = 1/2.$$

The equilibrium is depicted in the following figure.



6)

(a) $\theta = x_1 + x_2$ is the "common value". The crucial thing here is that each player knows his signal or type x_i but not the opponent's; however, as soon as he wins the auction, he discovers something disappointing about the other player's valuation: that it wasn't high enough for him to bid more and win! Each player must foresee this and incorporate it into his problem of defining his bid.

The ex-ante payoff of bidder: with type x_i if he wins is not $E(\theta) - b_i$, but $E(\theta|b_i > b_j) - b_i = x_i + E(x_j|b_i > b_j) - b_i$ (1)

Now, if agent j is using a linear bid (that can only be linear in the information available to him - namely x_j and not θ), then $b_j = a_j + c_j x_j$ and we can re-write (1) as:

$$x_i + E(x_j|b_i > a_j + c_j x_j) - b_i = x_i + E(x_j < \frac{b_i - a_j}{c_j}) - b_i = x_i + \frac{b_i - a_j}{2c_j} - b_i$$

As $E(x_j|b_i > a_j + c_j x_j) = \frac{b_i - a_j}{2c_j}$.

The probability of winning is $P(b_i > b_j) = P(b_i > a_j + c_j x_j) = P(x_j < \frac{b_i - a_j}{c_j}) = \frac{b_i - a_j}{c_j}$.

If he loses, he gets 0 and a tie happens with probability 0, so bidder i sets b_i so as to:

$$\max_{b_i} \frac{b_i - a_j}{c_j} \left(x_i + \frac{b_i - a_j}{2c_j} - b_i \right)$$

Notice that this equals $P(b_i > b_j) E(\theta - b_i | b_i > b_j)$.

The FOC yields:

$$\frac{1}{c_j}\left(x_i+\frac{b_i-a_j}{2c_j}-b_i\right)+\left(\frac{1}{2c_j}-1\right)\frac{b_i-a_j}{c_j}=0\Leftrightarrow$$
$$\Leftrightarrow \quad b_i=\frac{c_j}{2c_j-1}x_i+a_j\frac{c_j-1}{2c_j-1}$$

So, we can write $b_i = a_i + c_i x_i$, where $a_i = a_j \frac{c_j-1}{2c_j-1}$ and $c_i = \frac{c_j}{2c_j-1}$. But since the problem is symmetric, we have $c_i = c_j = 1$ and $a_i = a_j = 0$. Therefore, the equilibrium strategies are $b_i = x_i$ and $b_j = x_j$.

(b) Without assuming linearity, we can simply assume that the bid is strictly increasing and differentiable - $b_j = b^*(x_j)$ where $b^{*\prime} > 0$.

We follow the same steps as in a). The ex-ante payoff of bidder i with type x_i if he wins is: $x_i + E(x_j|b_i > b_j) - b_i = x_i + E(x_j|b_i > b^*(x_j)) - b_i$ (2)

Since $b^*(.)$ is strictly increasing, let $\phi(.)$ be the inverse of $b^*(.)$. We can re-write (2) as:

$$x_i+\underbrace{E(x_j|x_j<\phi(b_i))}_{\phi(b_i)/2}-b_i$$

The probability of winning is $P(b_i > b_j) = P(b_i > b^*(x_j)) = P(x_j < \phi(b_i)) = \phi(b_i)$. So, the bidder solves:

$$\max_{b_i}\phi(b_i)\left[x_i+\frac{\phi(b_i)}{2}-b_i\right]$$

The FOC yields:

$$\phi'(b_i)\left(x_i+\frac{\phi(b_i)}{2}-b_i\right)+\left(\frac{\phi'(b_i)}{2}-1\right)\phi(b_i)=0\Leftrightarrow$$
$$\Leftrightarrow \quad \phi' x_i+\phi\phi'-\phi' b_i-\phi=0$$

But $x_i = \phi(b_i)$ by symmetry and $2\phi\phi' - \phi' b_i - \phi = 0$ which is a differential equation for which one solution is $\phi(b) = kb + c$, with k determined by

$$\begin{aligned}
&2(kb+c)k-kb-kb-c=0\Leftrightarrow\\
\Leftrightarrow\ &2k^2b+2kc-2kb-c=0\Leftrightarrow\\
\Leftrightarrow\ &2kb(k-1)+c(2k-1)=0\Rightarrow\\
k\ =\ &1 \text{ and } c=0 \text{ and } \phi(b)=b
\end{aligned}$$

Therefore, $b^*(x_i) = x_i$.