Microeconomic Theory II
PS 4

1. This is a standard hidden-information agency problem.

That is, the agent's utility is $s + u(x,\theta) = s + (\theta+1)x - \frac{1}{2}x^2$ and the firm's (principal's) is $-s$. To employ the standard framework fully,

we need to ensure that the Spence-Mirrlees condition is satisfied. In this problem, it is sufficient to examine the cross-partial derivative of u:

$\frac{\partial^2 u}{\partial x \partial \theta} = \frac{\partial}{\partial x}x = 1 > 0$, so Spence-Mirrlees is satisfied. In this problem, surplus is $u + w = \Omega = (\theta+1)x - \frac{1}{2}x^2$, $F(\theta) = \theta$, and $f(\theta) = 1$, so virtual surplus is $\Sigma(x,\theta) = (\theta+1)x - \frac{1}{2}x^2 - \frac{1-\theta}{1} \cdot \frac{\partial u}{\partial \theta} = (\theta+1)x - \frac{1}{2}x^2 - (1-\theta)x = 2\theta x - \frac{1}{2}x^2$,

so the optimal $x^*(\theta) = \arg\max \Sigma(x,\theta)$. The first-order condition is $2\theta - x = 0$, so $x^*(\theta) = 2\theta$.

Also, $s(\theta) = -[(\theta+1)x^*(\theta) - \frac{1}{2}[x^*(\theta)]^2 + \int_0^\theta \frac{\partial u}{\partial \theta}(x^*(t),t)dt = -2\theta + \theta^2$

Finally, $p(x) = -s(x^{*-1}[x^*(\theta)]).x^{*-1}(x) = x/2$, so $p(x) = x - x^2/4$.

2.

(a) Player i´s payoff function is:

$$u_i(b_i, b_{-i}, \theta_i, \theta_{-i}) = \begin{cases} \theta_i - b_i & i \text{ submits winning bid} \\ -b_i & i \text{ submits losing bid} \end{cases}$$

The set of strategies $(b_j(\theta_j)\ \forall j = 1, ..., I)$ constitutes a BNE here, if for each $v_i \in [0,1]$, $b_i(v_i)$ solves:

$$\max_{b_i} (\theta_i - b_i) \Pr[b_i > b_j(\theta_j)]^{I-1} + (-b_i)\left[1 - \Pr[b_i > b_j(\theta_j)]^{I-1}\right]$$
$$\Leftrightarrow \max_{b_i} -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1}$$

To solve for a BNE, suppose that player j adopts the strategy $b(.)$ and assume that $b(.)$ is strictly increasing and differentiable. Then for a given realization of θ_i , player *i´s* optimal bid solves:

$$\max_{b_i} -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1}$$

Let $b_i^{-1}(b_j) = b^{-1}(b(\theta_j)) = \theta_j$ the valuation that player j must have in order to be bidding b_j. Since $\theta_j \sim U[0,1]$ we have:

$$\begin{aligned} & -b_i + \theta_i \Pr[b_i > b_j(\theta_j)]^{I-1} \\ = \quad & -b_i + \theta_i \Pr\left[b^{-1}(b_i) > b^{-1}(b(\theta_j))\right]^{I-1} \\ = \quad & -b_i + \theta_i \left[\frac{b^{-1}(b_i)}{1-0}\right]^{I-1} \end{aligned}$$

Thus, the first order condition for player *i´s* optimization problem is:

$$-1 + \theta_i (I-1)\left[\frac{db^{-1}(b_i)}{db_i}\right]\left[b^{-1}(b_i)\right]^{I-2} = 0$$

The first order condition (III) is an implicit equation for bidder $i\acute{s}$ best response to the strategy $b(.)$ played by bidder j, given that bidder $i\acute{s}$ valuation has been realized as θ_i . If we are looking for a symmetric BNE, we require that both bidders play the same strategy in equilibrium. Since, therefore, bidder j plays the strategy $b(.)$, this must be also playedby bidder i, in equilibrium. Hence, we require that $b(.)$ is player $i\acute{s}$ best response to $b(.)$ by player j. In other words, $b({}_i)$ must satisfy the first order condition (II): that is, for each of bidder $i\acute{s}$ positive valuations, she does not wish to deviate from bidding according to the schedule $b(.)$, given that player j bids according to the same schedule.

To impose this requirement, we substitute $b_i = b(\theta_i)$ into (iii):

$$\begin{aligned}
-1 + \theta_i (I-1) \left[\frac{db^{-1}(b_i)}{db_i}\right] \left[b^{-1}(b_i)\right]^{I-2} &= 0 \Leftrightarrow \theta_i (I-1) \left[\frac{d\theta_i}{db_i}\right] \theta_i^{I-2} = 1 \\
\theta_i^{I-1} (I-1) \left[\frac{db_i}{d\theta_i}\right] &= 1
\end{aligned}$$

Our last equation must be viewed as a first-order differential equation that the function $b(.)$ must satisfy. Clearly, however, if this is to be satisfy for any values of θ_i $\forall i$, it should be so for $\theta_i = \theta$ $\forall i$. We now have:

$$\frac{\theta^{I-1}}{b\acute{}(\theta)}(I-1) = 1 \Leftrightarrow b\acute{}(\theta) = (I-1)\,\theta^{I-1} \Leftrightarrow b(\theta) = \frac{I-1}{I}\theta^I + c$$

To eliminate c we need a boundary condition. Fortunately, simple economic reasoning provides us with one: no player should bid more than his/her valuation. Thus, we require $b(\theta_i) \leq \theta_i$ $\forall \theta_i \in [0,1]$. In particular, we require $b(0) \leq 0$. Since bids are constrained to be non-negative, this implies that $b(0) = 0$. Hence, $c = 0$ and our proposed BNE solution is that each bidder submits a bid according to the schedule:

$$b(\theta_i) = \frac{I-1}{I}\theta_i^I$$

(b) Using the Revenue-Equivalence theorem, we note that:

1. Both auctions can be viewed as incentive-compatible, direct-selling mechanisms.

2. In both auctions, the probability assignment function is the same since the object is assigned, in equilibrium, to the player with the highest valuation.

This is due the fact that, in both auctions, the players. equilibrium bidding strategies are strictly increasing in the players own valuations.

3. In both auctions, a bidder with zero valuation receives an equilibrium expected payoff of zero. Therefore, he is clearly indifferent between the two auction mechanisms.

(1)-(3) suffice for the theorem to apply. Consequently, the expected revenue to the seller ought to be the same between the sealed-bid second price auction and the first-price all-pay auction.

(c) Optimal asymmetric auction
- risk neutrality and independent private values
$\theta_1\tilde{}U[0,10]; \theta_2\tilde{}U[0,1]; \theta_3\tilde{}U[0,1]$
We can apply the same steps as in the lecture notes to get the expected revenue to seller:

$\int_0^{10} \partial\theta_1 \int_0^1 \partial\theta_2 \int_0^1 \partial\theta_3 [P_1(\theta_1,\theta_2,\theta_3)J_1(\theta_1) + P_2(\theta_1,\theta_2,\theta_3)J_2(\theta_2) + P_3(\theta_1,\theta_2,\theta_3)J_3(\theta_3)]\frac{1}{10}$
where the virtual valuations are:
$J_1(\theta_1) = 2\theta_1 - 10$; $J_2(\theta_2) = 2\theta_1 - 1$; $J_3(\theta_3) = 2\theta_3 - 1$;
All these are increasing in θ_i so the problem is regular $\rightarrow$ the optimal auction is characterized by:

Which gives:
$P_1(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_1 > 5$ and $\theta_1 > 4.5 + \theta_2$ and $\theta_1 > 4.5 + \theta_3$
$P_2(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_2 > 0.5$ and $\theta_2 > \theta_3$ and $\theta_2 > \theta_1 - 4.5$
$P_3(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_3 > 0.5$ and $\theta_3 > \theta_2$ and $\theta_3 > \theta_1 - 4.5$
We know from class that
$T_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} X_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i X_i(\theta_i)$

Rewriting it with the notation of the exercise ($p_i = X_i$ and $P_i = x_i$) and using the fact that the utility of the worst type must be zero, we get (using integration by parts):
$T_i(\theta_i) = \int_{0_i}^{\theta_i} p_i(\widetilde{\theta}_i)d\widetilde{\theta}_i - \theta_i p_i(\theta_i) = -\int_{0_i}^{\theta_i} \widetilde{\theta}_i dp(\widetilde{\theta}_i)$
So the agent expects to pay $t_i(\theta_i) = \int_{0_i}^{\theta_i} \widetilde{\theta}_i dp(\widetilde{\theta}_i)$
(where $p_i(\theta_i) = E_{\theta_{-i}}[P_i(\theta)]$).
One way to replicate this optimal auction by a standard-like auction:
- consider a Vickrey second price auction with a reserve price of 5 but where bidders 2 and 3 get a rebate of 4.5 if they win (this means that bidder 1 end up paying more than 5 if he wins). In this auction it is a dominant strategy to bid one´s valuation (+4.5 for bidders 2 and 3) $\rightarrow$ probabilities of winning and expected payment correspond to the ones found above.

Notice: An assumption all along is that the auctioners knows the distributions of types across bidders and who is who Bidders 2 and 3 receive preferential treatment here. This policy has a cost: sometimes the object will be awarded to someone else than the highest valuation bidder (non-efficiency) and the auctioneer will receive a relatively low payment in those cases.

However this bias against 1 forces him to bid higher than otherwise. The optimal auction strikes the best compromise between these two effects.

(d) $\theta_1\tilde{\ }U[1, 11]$
As above, we get:
$J_1(\theta_1) = 2\theta_1 - 11$; $J_2(\theta_2) = 2\theta_2 - 1$; $J_3(\theta_3) = 2\theta_3 - 1$
Which gives:
$P_1(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_1 > 5.5$ and $\theta_1 > 5 + \theta_2$ and $\theta_1 > 5 + \theta_3$
$P_2(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_2 > 0.5$ and $\theta_2 > \theta_3$ and $\theta_2 > \theta_1 - 5$
$P_3(\theta_1,\theta_2,\theta_3) = 1$ iff $\theta_3 > 0.5$ and $\theta_3 > \theta_2$ and $\theta_3 > \theta_1 - 5$
A similar 2nd price auction with minimum bid of 5.5 and rebate of 5 for bidders 2 and 3 is optimal

And this even though the auctioneer a priori knows that bidder 1's valuation is always greater than bidders 2 and 3's valuations.

3. (a) This is strategically equivalent to a sealed-bid second-price auction and it is a (weakly) dominant strategy for the buyer to bid her true valuation b. However, the problem is different for the seller who solves $\max_p prob(p \leq b)(p - s) = \max_p(1 - p)(p - s)$ yielding $p = s/2 + 1/2$.

(b) Let b denote the buyer's valuation that follows a $U[0, 1]$ distribution. Let $T(b, s)$ denote the (possibly negative) transfer from the seller to the buyer and let $X(b, s)$ denote the decision function (probability of the object being sold).
Let $x_B(b) = E_s(X(b, s))$ and $t_B(b) = E_s(T(b, s))$.

The interim utility of the buyer is $u_B = t_B + bx_B$

(i) IC

For the buyer's IC, from $U_B(\widehat{b}, b) = t_B(\widehat{b}) + bx_B(\widehat{b})$ we can write the value function $U_B(b) \equiv U_B(b, b)$. Using the value function and applying the envelope theorem (or just incorporating the FOC), we have that $\frac{dU_B}{db}(b) = \frac{\partial u_B}{\partial b}(b)$ and

$$\frac{dU_B}{db}(b) = x_B(b).$$

Since $\frac{d}{db}(\frac{\frac{\partial u_B}{\partial x_B}}{\frac{\partial u_B}{\partial t_B}}) = 1 > 0$, CS^+ holds. Therefore, we need $x_B(b)$ to be nondecreasing in any implementable contract.

$IC \Longleftrightarrow \begin{cases} \frac{dU_B}{db}(b) = x_B(b) \\ x_B(b) \text{ nondecreasing} \end{cases}$

We can now rewrite the profits of the buyer as the sum of the profits of the "worst type" and an integral. Integrating $\frac{dU_B}{db}(b) = x_B(b)$ from $\underline{b}= 0$ to b yields $U_B(b) = U_B(0) + \int_0^b x_B(\widetilde{b})d\widetilde{b}$. Since $U_B(b) = t_B(b) + bx_B(b)$, we can write $t_B(b) = U_B(0) + \int_0^b x_B(\widetilde{b})d\widetilde{b} - bx_B(b)$.

(ii) IR just requires that $U_B(0) \geq 0$. Since the seller wants to minimize the transfer, she will want to set $U_B(0) = 0$.

The seller wants to maximize $E_sE_b(-X(b, s)s-T(b, s)) = E_sE_b(-X(b, s)s)-E_b[\int_0^b x_B(\widetilde{b}_i)d\widetilde{b}_i- bx_B(b)]$ (subject to the monotonicity constraint).

Since

$$E_b\left[\int_0^b x_B(\widetilde{b})d\widetilde{b}\right] = \int_0^1 \int_0^b x_B(\widetilde{b})d\widetilde{b}db,$$

integration by parts yields

$$E_b\left[\int_0^b x_B(\widetilde{b})d\widetilde{b}\right] = E_b\left[x_B(b).(1-b)\right]$$

The simplified problem is then:

$$\max_{\{X(b,s)\}} E_sE_b(-X(b, s)s - X(b, s)(1 - 2b)) = \max_{\{X(b,s)\}} E_sE_b[X(b, s)(2b - 1 - s))$$

The optimal mechanism is the one that sets $X(b, s) = 1$ if $1 + s \leq 2b$ and 0 otherwise.

However, this mechanism would not be ex post efficient, because the object might not be sold in cases where the buyer's valuation exceeds the seller's.

4. (a) Since both q and θ are observable, the principal solves (for each θ):

$$\max_{q,t} \int_0^q p(s)ds - \theta q - (1 - \alpha)\pi$$

$$\text{s.t. } \pi \geq 0$$

At the optimum, the principal will give 0 profits to the firm i.e. $\pi = 0 \Leftrightarrow$

$$t + qp(q) - \theta q = 0 \Leftrightarrow t = \theta q - qp(q)$$

and the problem is simplified to:

$$\max_q \int_0^q p(s)ds - \theta q$$

yielding the FOC $p(q) = \theta$, which implicitly defines $q(\theta)$: for each θ, q will be set so that the corresponding price equals marginal cost.

(b) From here on, we need to rewrite the problem so that C instead of q represents the firm's decision.

Since $C = \theta q$, we have $q = \frac{C}{\theta}$ and firm profit is $\pi = \frac{C}{\theta} p(\frac{C}{\theta}) + t - C$

So $\pi(\widehat{\theta}, \theta) = \frac{C(\widehat{\theta})}{\theta} p(\frac{C(\widehat{\theta})}{\theta}) + t(\widehat{\theta}) - C(\widehat{\theta})$ and the FOC of the problem $\max_{\widehat{\theta}} \pi(\widehat{\theta}, \theta)$ must be met for $\widehat{\theta} = \theta$ for the "local IC" to hold.

We then have

$$\frac{\partial C}{\partial \theta}(\theta) \left[\frac{1}{\theta} p(\frac{C(\theta)}{\theta}) - 1 \right] + \frac{C(\theta)}{\theta^2} p'(\frac{C(\theta)}{\theta}) + \frac{\partial t}{\partial \theta}(\theta) = 0.$$

Let $\pi(\theta) \equiv \pi(\theta, \theta)$. Using the value function and incorporating the FOC (or just applying the envelope theorem), we have that $(\theta) = \frac{\partial \pi}{\partial \theta}(\theta)$ and

$$\frac{d\pi}{d\theta}(\theta) = -\frac{C(\theta)}{\theta^2} p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta} p'(\frac{C(\theta)}{\theta})(-\frac{C(\theta)}{\theta^2}) = -\frac{C(\theta)}{\theta^2} \left[p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta} p'(\frac{C(\theta)}{\theta}) \right].$$

Note that $p(\frac{C}{\theta}) + \frac{C}{\theta} p'(\frac{C}{\theta})$ is just marginal revenue which is positive by assumption and therefore $\frac{d\pi}{d\theta}(\theta) < 0$.

(c) CS^+ means $\frac{d}{d\theta}(\frac{\frac{\partial \pi}{\partial C}}{\frac{\partial \pi}{\partial t}}) > 0$

(CS^-) $\qquad (<)$

$$\begin{aligned} \frac{\partial \pi}{\partial C} &= \frac{1}{\theta} p(\frac{C}{\theta}) + \frac{C}{\theta} p'(\frac{C}{\theta}) \frac{1}{\theta} - 1 \\ \frac{\partial \pi}{\partial t} &= 1 \end{aligned}$$

Therefore,

$$\frac{d}{d\theta}(\frac{1}{\theta} p(\frac{C}{\theta}) + \frac{C}{\theta} p'(\frac{C}{\theta}) \frac{1}{\theta} - 1) = -\frac{1}{\theta^2}(p(\frac{C}{\theta}) + \frac{C}{\theta} p'(\frac{C}{\theta})) + \frac{1}{\theta}(-\frac{2C}{\theta^2} p'(\frac{C}{\theta}) - \frac{C^2}{\theta^3} p''(\frac{C}{\theta})).$$

Note that $p(\frac{C}{\theta}) + \frac{C}{\theta} p'(\frac{C}{\theta})$ is just marginal revenue and $2p'(\frac{C}{\theta}) + \frac{C}{\theta} p''(\frac{C}{\theta})$ is $\frac{dMR}{dq}(\frac{C}{\theta})$

So $\frac{d}{d\theta}(\frac{\frac{\partial \pi}{\partial C}}{\frac{\partial \pi}{\partial t}}) = -\frac{1}{\theta^2}(MR + q\frac{dMR}{dq})$

Therefore, CS^+ means $MR + q\frac{dMR}{dq} < 0 \Leftrightarrow 1 + \frac{q}{MR}\frac{dMR}{dq} < 0 \Leftrightarrow \varepsilon_{MR,q} > 1$,

where $\varepsilon_{MR,q}$ is the elasticity of marginal revenue with respect to quantity. It is this condition that implies that any implementable contract must have the realized total cost be a non-decreasing function of type.

(d) We have $\frac{d\pi}{d\theta}(\theta) < 0$ from b). Therefore, the "worst type" is type $\overline{\theta}$. Integrating from θ to $\overline{\theta}$ both sides of the equation

$$\frac{d\pi}{d\theta}(\theta) = -\frac{C(\theta)}{\theta^2}\left[p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta}p'(\frac{C(\theta)}{\theta})\right]$$

and simplifying we get:

$$\pi(\theta) = \pi(\overline{\theta}) + \int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta}$$

(e) Here we need to solve the principal's problem. Given that $\pi(\theta)$ enters the principal's objective function with a negative sign, the principal will want to set $t(\overline{\theta})$ so that $\pi(\overline{\theta}) = 0$ (IR is binding for the "worst type").

At the optimum, we have $\pi(\theta) = \int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta}$ (and $t(\theta) = \pi(\theta) + C(\theta) - \frac{C(\theta)}{\theta}p(\frac{C(\theta)}{\theta})$).

Replacing $\pi(\theta)$ in the regulator's objective function (and ignoring the monotonicity constraint), the simplified problem is:

$$\max_{C(\theta)} E_\theta\left[\int_0^{\frac{C(\theta)}{\theta}} p(s)ds - C(\theta) - (1-\alpha)\int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta}\right]$$

Since

$$E_\theta\left[\int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta}\right] = \int_{\underline{\theta}}^{\overline{\theta}}\int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta} f(\theta)d\theta,$$

integration by parts yields

$$E_\theta\left[\int_{\theta}^{\overline{\theta}} \frac{C(\widetilde{\theta})}{\widetilde{\theta}^2}\left[p(\frac{C(\widetilde{\theta})}{\widetilde{\theta}}) + \frac{C(\widetilde{\theta})}{\widetilde{\theta}}p'(\frac{C(\widetilde{\theta})}{\widetilde{\theta}})\right] d\widetilde{\theta}\right] = E_\theta\left[\frac{C(\theta)}{\theta^2}\frac{F(\theta)}{f(\theta)}\left[p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta}p'(\frac{C(\theta)}{\theta})\right]\right]$$

The principal's problem is then

$$\max_{C(\theta)} E_\theta\left[\int_0^{\frac{C(\theta)}{\theta}} p(s)ds - C(\theta) - (1-\alpha)\frac{C(\theta)}{\theta^2}\frac{F(\theta)}{f(\theta)}\left[p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta}p'(\frac{C(\theta)}{\theta})\right]\right]$$

Pointwise differentiation yields:

$$\frac{1}{\theta}p(\frac{C(\theta)}{\theta}) - 1 - (1-\alpha)\frac{1}{\theta^2}\frac{F(\theta)}{f(\theta)}\left[\underbrace{p(\frac{C(\theta)}{\theta}) + \frac{C(\theta)}{\theta}p'(\frac{C(\theta)}{\theta})}_{MR}\right] - (1-\alpha)\frac{C(\theta)}{\theta^2}\frac{F(\theta)}{f(\theta)}\left[\frac{dMR}{dq}\frac{dq}{dC}\right] = 0$$

and

$$\frac{1}{\theta}p(\frac{C(\theta)}{\theta}) - 1 - (1-\alpha)\frac{1}{\theta^2}\frac{F(\theta)}{f(\theta)}\left(MR + \frac{C(\theta)}{\theta}\frac{dMR}{dq}\right) = 0 \Longleftrightarrow$$

$$p(\frac{C(\theta)}{\theta}) = \theta + (1-\alpha)\frac{1}{\theta}\frac{F(\theta)}{f(\theta)}\left(MR + q\frac{dMR}{dq}\right)$$

Therefore, $p(q) < \theta$ if $(1-\alpha)\frac{1}{\theta}\frac{F(\theta)}{f(\theta)}\left(MR + q\frac{dMR}{dq}\right) < 0 \Leftrightarrow \varepsilon_{MR,q} > 1$

(f) Once the optimal type-dependent contracts have been determined, the principal may implement them by announcing:

$$t(C) = \begin{cases} t(\theta) \text{ if there is a } \theta \text{ s.t. } C = C(\theta) \\ -\infty \text{ otherwise} \end{cases}$$

5. (a) With Budget Balance but no Individual Rationality constraints, we can use the AGV mechanism to achieve efficiency.

- the decision function $x^*(\widehat{\theta})$ solves the social surplus maximization problem

$$\max_x \sum_{i=1}^{I} v_i(x,\theta_i) + v_0(x,\theta) = \max_x \sum_{i=1}^{I} \theta_i x - cx \Rightarrow x^*(\theta) = \begin{cases} 1 \text{if } \sum_i \theta_i \geq c \\ 0 \text{if } \sum_i \theta_i < c \end{cases}$$

- each agent's transfer is:

$$t_i(\widehat{\theta}_i) = E_{\theta_{-i}}[\sum_{j\neq i} \theta_j x^*(\widehat{\theta}_i, \theta_{-i}) - cx^*(\widehat{\theta}_i, \theta_{-i})] + \tau_i(\widehat{\theta}_{-i})$$

The proof that we have Bayesian IC is straightforward. BB is also met if we set $\tau_i(\widehat{\theta}_{-i}) = -\sum_{j\neq i} \frac{E_{\theta_{-j}}\left(\sum_{k\neq j}\theta_k x^*(\widehat{\theta}_j,\theta_{-j})\right)}{I-1} + \sum_{j\neq i}\frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j,\theta_{-j}))}{I}$
so that

$$\sum_i E_{\theta_{-i}}[\sum_{j\neq i}\theta_j x^*(\widehat{\theta}_i,\theta_{-i}) - cx^*(\widehat{\theta}_i,\theta_{-i})] - \sum_i\sum_{j\neq i}\frac{E_{\theta_{-j}}\left(\sum_{k\neq j}\theta_k x^*(\widehat{\theta}_j,\theta_{-j})\right)}{I-1} + \sum_i\sum_{j\neq i}\frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j,\theta_{-j}))}{I}$$

$$= \quad -\sum_i E_{\theta_{-i}}(cx^*(\widehat{\theta}_i,\theta_{-i})) + \sum_i\sum_{j\neq i}\frac{E_{\theta_{-j}}(cx^*(\widehat{\theta}_j,\theta_{-j}))}{I} = -\frac{1}{I}\sum_i E_{\theta_{-i}}(cx^*(\widehat{\theta}_i,\theta_{-i})).$$

(b) Dropping BB, we can use the Groves-Clark mechanism to achieve dominant-strategy implementation and efficiency:

- the decision function $x^*(\widehat{\theta})$ again solves the social surplus maximization problem
- each agent's transfer is now $t_i(\widehat{\theta}) = \sum_{j\neq i}\widehat{\theta}_j x^*(\widehat{\theta}_i,\widehat{\theta}_{-i}) - cx^*(\widehat{\theta}_i,\widehat{\theta}_{-i}) + \tau_i(\widehat{\theta}_{-i})$

and the proof that it is a dominant strategy to report truthfully is straightforward.

IR would not be a problem then: with no BB, the owner would simply make arbitrarily large transfers (using the $\tau_i(\widehat{\theta}_{-i})$ component of the transfer) to ensure participation.

6. FT 7.10

a) Seller owns 1 unit of a good which she values at c which can take the values: $\underline{c}$ with prob $\frac{1}{2}$ or $\overline{c}$ with prob $\frac{1}{2}$ ($\underline{c} < \overline{c}$). Seeler´s valuation is private information to the seller.

Buyer´s valuation is $\overline{v}$ if $c = \overline{c}$ and $\underline{v}$ if $c = \underline{c}$ where $\overline{v} > \overline{c}$ and $\underline{v} > \underline{c}$ and $\frac{\overline{v}+\underline{v}}{2} < \overline{c}$ ($\Rightarrow \overline{c} > \underline{v}$) ,i.e., $\underline{c} < \underline{v} < E(v) < \overline{c} < \overline{v}$

Show that efficiency is inconsistent with seller´s and buyer´s IR and IC

Efficiency in this case means that trade always occurs since there are always gains from trade. Denoting by $x(c)$ the probability of trade when the seller´s valuation is c, efficiency implies:

$$x(\underline{c}) = x(\overline{c}) = 1 \quad (1)$$

IR: letting $t(c)$ denote the tranfer from buyer to seller when the seller has valuation c, for the seller (who knows his valuation), IR means:

$$t(\underline{c}) - x(\underline{c})\,\underline{c} \geq 0 \quad (2)$$

$$t(\overline{c}) - x(\overline{c})\,\overline{c} \geq 0 \quad (3)$$

For the buyer, who does not know the seller´s valuation, IR means

$$\frac{1}{2}\left(\underline{v}x(\underline{c}) - t(\underline{c})\right) + \frac{1}{2}\left(\overline{v}x(\overline{c}) - t(\overline{c})\right) \geq 0 \quad (4)$$

(ex-ante expected utility associated with the mechanism $\{x(\overline{c}), x(\underline{c}), t(\overline{c}), t(\underline{c})\}$)

IC: since only the seller has private information, we only have an IC constraint for the seller:

$$t(\underline{c}) - x(\underline{c})\,\underline{c} \geq t(\overline{c}) - x(\overline{c})\,\underline{c} \quad \text{IC for type } \underline{c} \quad (5)$$

$$t(\overline{c}) - x(\overline{c})\,\overline{c} \geq t(\underline{c}) - x(\underline{c})\,\overline{c} \quad \text{IC for type } \overline{c} \quad (6)$$

Incorporating (1) into (5) and (6) yields
$t(\underline{c}) \geq t(\overline{c})$ and $t(\overline{c}) \geq t(\underline{c}) \Rightarrow t(\overline{c}) = t(\underline{c}) = t$ (7)
Using (1) and (7) in the buyer´s IR(4) and the seller´s IR(3), yields:
$t \leq \frac{1}{2}\underline{v}+\frac{1}{2}\overline{v}$ and $t \geq \overline{c} \Rightarrow \frac{1}{2}\underline{v}+\frac{1}{2}\overline{v} \geq \overline{c}$ which contradicts the assumption $\frac{\overline{v}+\underline{v}}{2} < \overline{c}$
Therefore, efficiency is incompatible with seller´s and buyer´s IR and IC.

The Myerson Satterthwaite theorem cannot be applied here because we always have gains from trade (there is perfect correlation between the buyer´s type and the seller´s type and probability 1 of gains from trade since $\overline{v} > \overline{c}$ and $\underline{v}>\underline{c}$), from an interval with a strictly positive density.

Even with a continum of buyers and sellers, where buyers are homogeneous and the qualities are independent across sellers, the incompatibility between efficiency and IC\IR still carries over (the exact same conditions must be hold and they do not, under the assumptions). So, even with an uncountably infinite number of agents, although each agent´s private information is only a small fraction of the information that is not known by the others agents, it is still relevant for all the other agents and therefore prevents an efficient outcome.

b) Now each seller has private information that is relevant to a single buyer instead of to all buyers.

Duplex:
- 1st floor: agent F owns a smoke alarm that he values at: $\overline{c}$ if he smokes and $\underline{c}$ if not
- 2nd floor: agent S does not know if F smokes and values the somke alarm at: $\overline{v}$ if F smokes and $\underline{v}$ if not.

where $\overline{v} > \overline{c}, \underline{v} > \underline{c}$

The probability of smoking is $\frac{1}{2}$ for F, S does not smoke and that is commom knowledge.

Construct an efficient, IC and IR mechanism for trade in an economy with a continuum of such duplexes.

With only a duplex, since S values the smoke alarm more than F whether F smokes or not, the only efficient outcome is to transfer the alarm frm F to S - condition (1) from a) must be satisfied: $x(\underline{c}) = x(\overline{c}) = 1$ where $x(c)$ is the probability of the trade given the seller´s valuation c; but we know that this is incompatible with IR\IC

With r duplexes, we have the possibility of trade not only between F and S in the same duplex, but also between any two agents (e.g. S_1 and B_3 or S_7 and B_2, i.e., the first floor resident of one duplex and the second floor resident of a different duplex). Notice, however, that the valuation of a second-floor resident depends only on the valuation of the first-floor resident of the same duplex (i.e., it depends on whether he smokes or not) but not on the valuations of all the residents of the other duplexes. So, onlike the case in a) where the buyer´s valuation was directly linked with the quality of the good reflected on the seller´s valuation (and the seller´s valuation was therefore relevant for any possible buyer), here we have that the buyer´s valuation is not related to the quality of the smoke alarm but rather to the valuation of the first floor resident of the buyer´s duplex (and this valuation is therefore only relevant for that particular buyer and each F will become informationally small as we increase the size of the economy)

So, with a continuum of duplexes, we can achieve efficiency with an IR\IC mechanism: the mechanism specifies that all agents F who smoke should keep the alarm and all the agents S who live in a duplex with a smoker F should get the alarm from all agents F who do not smoke at a fixed price $t \in (\underline{v}, \overline{c})$ (i.e., the alarm is either sold at this price or not sold at all)

Efficiency in this setting means that all the agents that value the smoke alarm the most should get it, and given the assumptions $\underline{c} < \underline{v} << \overline{c} < \overline{v}$, and given that the probability of the proportion of smokers being $\frac{1}{2}$ is one (we have an infinite number of agents), it is efficient to let all agents with valuations $\overline{v}$ and $\overline{c}$ have the alarm. So the mechanism is efficient since it specifies that all the residents of the duplexes where F smokes have a smoke alarm.

The mechanism is incentive compatible: every agent F announces whether he smokes or not and will want to do so trhthfully, if F smokes and announces he smokes, he keeps the smoke alarm and won´t trade but if he announces he does not smoke, he sells the smoke alarm at price t to an agent S with valuation $\overline{v}$; Since $t < \overline{c}$, $0 > t - \overline{c}$ and F would rather say he smokes and get 0. If F does not smoke and announces so, he gets $t - \underline{c} > 0$ (he sells the alarm) and he would rather announce truthfully than to announce he smokes and get 0.

Also the mechanism is individually rational for both types of agent F (again since $t > \underline{c}$) and for all agents S (since $\overline{v} > t$).

So, in this case all the potential gains from trade are realized.